Institute of Theoretical Physics
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TENTAMEN GENERAL RELATIVITY

wednesday 25-6-1997, 14.00-17.00, room 11.22

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

The Robertson-Walker metric for $k = 1$ can be written in the form (we take $c = 1$)

$$ds^2 = dt^2 - R(t)^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}\,. \tag{1}$$

For the energy-momentum tensor of a perfect fluid the Einstein equations lead to the following relations between the function $R(t)$, the mass density $\rho(t)$ and the pressure $p(t)$:

$$\frac{(\dot{R})^2 + 1}{R^2} = \tfrac{1}{3}\kappa\rho\,, \tag{2}$$

$$\dot{\rho} + 3(p+\rho)\frac{\dot{R}}{R} = 0\,. \tag{3}$$

The dot indicates a differentiation with respect to t and $\kappa = 8\pi G$ (G is Newton's constant).

We consider the situation of a Friedmann universe with ultra-relativistic matter, i.e. $p = \frac{1}{3}\rho$.

(1.1) Show that ρR^4 is constant.

(1.2) Determine R as a function of t. Take as boundary condition that $R = 0$ at $t = 0$. Let ρ_0 and R_0 be the values of the functions ρ and R at the time $t = t_0$. Show that this universe has a finite lifetime and determine this lifetime in terms of ρ_0, R_0 and Newton's constant G.

(1.3) Determine the orbit of light rays with $\dot{\theta} = \dot{\phi} = 0$. Hint: the following integral is needed:

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x \, . \tag{4}$$

(1.4) A light ray is emitted at the origin of the univere (i.e. $R = t = 0$) from a point with coordinate $\chi = 0$. What is the value of the coordinate χ of the light ray if the value of R has become again equal to zero? Has the light ray then travelled through the whole universe?

Question 2

The Schwarzschild metric (we take $c = 1$)

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

leads for constant r and θ to the following geodesic equations

$$\begin{aligned} (1 - \frac{2m}{r})\dot{t} &= k \, , & (6) \\ r^2\dot{\phi} &= h \, , & (7) \\ \frac{m}{r^2}(\dot{t})^2 - r(\dot{\phi})^2 &= 0 \, , & (8) \end{aligned}$$

with k and h constant. The dot $\cdot$ indicates differentation with respect to the parameter u of the geodesic.

(2.1) A light-ray follows a geodesic around a black hole for constant $r = r_0$ and $\theta = \pi/2$. Determine the value of r_0.

(2.2) Determine the coordinate time T that the light ray needs to complete one circular orbit.

(2.3) Determine the circumference of the circle with $r = r_0$.

(2.4) An observer, in a rocket, finds himself at the point with constant coordinates $r = r_0, \phi = 0, \theta = \pi/2$. What is the time that evolves at the watch of this observer during one revolution of the light-ray? What is according to the observer the velocity of light?

(2.5) Using his rocket the observer now follows the orbit of the light-ray, see question (2.1), with a constant angular velocity $\omega = d\phi/d\tau$, where τ is the eigentime of the observer. At the time $t = t_0$ the observer stops his rocket. Determine the acceleration in the r-direction that the observer then gets. Hint: Use that for the Schwarzschild metric

$$\Gamma^r_{tt} = \frac{m}{r^2}(1 - \frac{2m}{r}), \ \Gamma^r_{t\phi} = 0, \ \Gamma^r_{\phi\phi} = -r(1 - \frac{2m}{r}). \tag{9}$$

Question 3

The Maxwell equations in a four-dimensional curved space can be written in the form

$$\nabla_\nu F^{\mu\nu} = j^\mu, \tag{10}$$
$$\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0. \tag{11}$$

Here F is the anti-symmetric field-strenght tensor and j the current. The covariant derivatives are with respect to the metric connection.

(3.1) Show that equation (11) for $F_{\mu\nu}$ is equivalent to

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0. \tag{12}$$

Hint: The covariant derivative of a contravariant vector V^μ and a covariant vector W_μ are given by

$$\begin{aligned} \nabla_\nu V^\mu &= \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda, \\ \nabla_\nu W_\mu &= \partial_\nu W_\mu - \Gamma^\lambda_{\mu\nu} W_\lambda. \end{aligned} \tag{13}$$

(3.2) We write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{14}$$

where A_μ is a covariant vector. Show that the $F_{\mu\nu}$ defined in this way is a covariant tensor by showing that

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{15}$$

Show that this $F_{\mu\nu}$ is a solution of equation (11).

(3.3) Show that equation (10) may alternatively be written as

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = \sqrt{-g}j^\mu \,, \tag{16}$$

with $g = \det\,(g_{\mu\nu})$. Hint: the metric connection is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\} \,. \tag{17}$$

Furthermore, we have the identity

$$\partial_\mu g = g g^{\rho\sigma}\partial_\mu g_{\rho\sigma} \,. \tag{18}$$

(3.4) Show that equation (10) implies that

$$\nabla_\mu j^\mu = 0 \,. \tag{19}$$